CSCI 341 Workshop 1

Automata and Languages

September 1, 2025

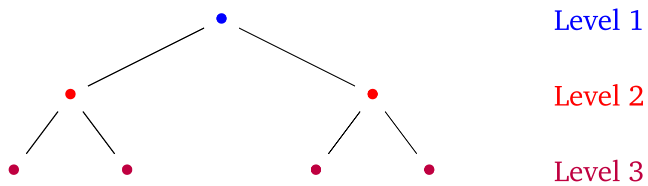
Theorem 0.1 (Ordinary Induction). *Let $S \subseteq \mathbb{N}$ be a set of natural numbers. If the following two statements are true:*

Base Case *The number $0 \in S$.*

Induction Step *If the number $n \in S$, then the number $n + 1 \in S$.*

then $S = \mathbb{N}$.

Problem 1 (Counting Trees). Recall that a complete binary tree is a binary tree in which every level of the tree is entirely full. For example, the following tree is a complete binary tree with three levels.



The complete tree with three layers has 7 nodes. Using *Ordinary Induction*, prove that the complete binary tree with n layers has $2^n - 1$ nodes. For example, $2^3 - 1 = 8 - 1 = 7$.

Problem 2 (*Bonus* Counting Prefixes). Given a word $w = a_0 a_1 \cdots a_{n-1}$, a *subword* of w is a consecutive sequence of indices $[k, k+1, k+2, \ldots, l-1]$ for $0 \leqslant k \leqslant n-1$. We often identify a subword with the word $u = a_k a_{k+1} \cdots a_{l-1}$, so that there exist two other subwords v_1 and v_2 such that $w = v_1 u v_2$.

$$\underbrace{a_0 a_1 a_2 \cdots a_{n-1}}_{w} = \underbrace{a_0 a_1 \cdots a_{k-1}}_{v_1} \underbrace{a_k a_{k+1} \cdots a_{l-1}}_{u} \underbrace{a_l a_{l+1} \cdots a_{n-1}}_{v_2}$$

For example, the subwords of aba are

$$\underbrace{[]}_{\varepsilon}, \underbrace{[0]}_{a}, \underbrace{[1]}_{b}, \underbrace{[2]}_{a}, \underbrace{[0,1]}_{ab}, \underbrace{[1,2]}_{ba}, \underbrace{[0,1,2]}_{aba}$$

of which there are 7. Using Ordinary Induction, prove that a word of length n has

$$\frac{n(n+1)+2}{2}$$

many subwords. *Hint: Every subword of the word wa is either a subword of w or a suffix of wa, i.e., ends with a. also, $(n+1)(n+2) = n^2 + 3n + 2$.*

Solution to Counting Trees. Let

$$S = \{n \mid \text{the complete binary tree with } n \text{ layers has } 2^n - 1 \text{ nodes}\}$$

Our goal is to prove that $S = \mathbb{N}$. We proceed by induction on n.

Base Case

Induction Hypothesis

Induction Step:

$\square$

Solution to Counting Subwords. We proceed by induction on n.

Base Case

Induction Hypothesis

Induction Step:

□

Theorem 0.2 (Induction on Words). *Let $L \subseteq A^*$ be a language. If the following two statements are true:*

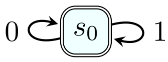
Base Case *The empty word $\varepsilon \in L$ is in the language.*

Induction Step *If $w \in L$, then for any $a \in A$, $wa \in L$.*

then $L = A^*$.

Problem 3 (Double-reversal). Given a word w, define w^{op} to be the *reversal* of the word, as follows: on the empty word, we define $\varepsilon^{\mathrm{op}} = \varepsilon$. Given a word $w \in A^*$ and a letter $a \in A$, we define $(wa)^{\mathrm{op}} = aw^{\mathrm{op}}$. Use Induction on Words to prove that for any word $w \in A^*$, $(w^{\mathrm{op}})^{\mathrm{op}} = w$.

Problem 4 (All-accepting). Let $A = \{0, 1\}$. Use Induction on Words to prove that the all-accepting automaton accepts every word form A. That is,

$\mathcal{A}_{\checkmark} =$ 

$\mathcal{L}(\mathcal{A}_{\checkmark}, s_0) = A^*$

Problem 5 (*Bonus* Double Double Reversal). Use Induction on Words to prove that for all $w, u \in A^*$, the reversal of their concatenation is the reversed concatenation of their reversals:

$$(wu)^{\mathrm{op}} = u^{\mathrm{op}} w^{\mathrm{op}}$$

Solution to Double Reversal. Let $L = \{w \mid (w^{\text{op}})^{\text{op}} = w\}$. The goal is to show that $L = A^*$. We proceed by induction on $w \in L$.

Base Case

Induction Hypothesis

Induction Step:

□

Solution to All-accepting. Let $L = \mathcal{L}(\mathcal{A}_{\checkmark}, s_0)$. The goal is to show that $L = A^*$. We proceed by induction on $w \in L$.

Base Case

Induction Hypothesis

Induction Step:

□

Solution to Double Double Reversal. This one is a bit trickier than Double Reversal, because there are two words involved in the statement. Interestingly, we only need to involve one of the words in the proof: Let

$$L = \{w \mid \text{for any word } u, (wu)^{\text{op}} = u^{\text{op}} w^{\text{op}}\}$$

The goal is to show that $L = A^*$. We proceed by induction on $w \in L$.

Base Case

Induction Hypothesis

Induction Step:

□